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The Boson Gas on a Cayley Tree

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We analyze the free boson gas on a Cayley tree using two alternative methods. The spectrum of the lattice Laplacian on a finite tree is obtained using a direct iterative method for solving the associated characteristic equation and also using a random walk representation for the corresponding fermion lattice gas. The existence of the thermodynamic limit for the pressure of the boson lattice gas is proven and it is shown that the model exhibits boson condensation into the ground state. The random walk representation is also used to derive an expression for the Bethe approximation to the infinite-volume spectrum. This spectrum turns out to be continuous instead of a dense point spectrum, but there is still boson condensation in this approximation.

KEY WORDS: Boson condensation; Cayley tree, random walk representation.

1. INTRODUCTION

Although some progress has been made in recent years, the occurrence of boson condensation in a true quantum mechanical model of an interacting boson gas is still an open problem. In order to make progress, simplifying assumptions have to be made. In ref. 1, only terms in the Hamiltonian that are diagonal in the occupation numbers were retained, while in ref. 12, a lattice model was considered with a mean-field-type Laplacian. In this article we aim to initiate a study of another lattice model obtained by simplifying the lattice to a Cayley tree. It is well known that statistical mechanical models on a Cayley tree are often simpler to analyze than

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models on a regular Bravais lattice and one may hope that the boson gas is no exception. We restrict ourselves here to a consideration of the free boson gas and hope to consider interacting models in a subsequent paper. In Section 2 we give a precise definition of the model and derive the finite-volume spectrum by a simple iterative procedure. In Section 3 an alternative derivation of the spectrum is presented using a random walk representation of the free fermion gas on the tree. The spectrum of the Hamiltonian can then be read off from the formula obtained for the finite-volume pressure. In this method the partition function of the fermion gas is written in terms of discrete-time walks on an augmented lattice, which can then be analyzed in terms of generating functions analogous to ref. 9. In Section 4 the free boson gas on the tree is analyzed. The usual boson gas exhibits boson condensation for dimensions $d \geqslant 3$,[7] and since the Cayley tree has in some sense an infinite dimension, one may expect boson condensation in this lattice model as well. We shall prove that this is indeed the case. In fact, we first prove generalized condensation in the sense of Girardeau[4] in Section 4, and then in Section 5 we prove more precisely that the condensation is entirely in the lowest energy state, i.e., $\rho_m = \rho_c$ in the terminology of ref. 2. In Section 6 we consider the Bethe approximation to the boson lattice gas pressure using the random walk representation. We find that it differs from the Cayley tree result in that the single-particle spectrum is continuous, as opposed to the spectrum of the Laplacian on the infinite Cayley tree, which is a dense point spectrum. Both exhibit boson condensation, however. The situation is therefore different from the Ising model, where the Bethe approximation exhibits a phase transition, whereas the model on a Cayley tree does not.[3]

2. DESCRIPTION OF THE MODEL AND ITS SPECTRUM

Let Γ_q be an infinite homogeneous tree of degree $q \geqslant 3$. We can choose an arbitrary point of the tree as a root point, which we denote as 0, and introduce a coordinate system on the tree by labeling the rest of the vertices of Γ_q with $(k; i_1,..., i_k)$, where $k = 1, 2,...$ is the distance from the root and $i_1 = 1, 2,..., q$ and $i_k = 1, 2,..., q-1$ $(k \geqslant 2)$ denote the branch to be taken at the first, respectively kth, junction. We also introduce the finite Cayley ball Γ_q^N of radius N about the root chosen: it is the finite subgraph of Γ_q spanning all the vertices $(k; i_1,..., i_k)$ with $k \leqslant N$. The set of vertices of Γ_q and Γ_q^N will be denoted by V_q and V_q^N, respectively. The number of vertices of Γ_q^N is given by

$$|V_q^N| = \frac{q(q-1)^N - 2}{q-2} \tag{2.1}$$

The Laplacian Δ on $l^2(V_q)$ is a bounded self-adjoint operator with matrix representation

$$\Delta_{(k;\, i_1,\dots,\, i_k),\,(l;\, j_1,\dots,\, j_l)} = \delta_{k,l+1} \prod_{r=1}^{l} \delta_{i_r, j_r} + \delta_{k+1,l} \prod_{r=1}^{k} \delta_{i_r, j_r} - q\delta_{k,l} \prod_{r=1}^{k} \delta_{i_r, j_r} \tag{2.2}$$

It is well known (ref. 8, p. 225) that the spectrum of $-\Delta$ is the set

$$\operatorname{spec}(-\Delta) = [q - 2(q-1)^{1/2}, q + 2(q-1)^{1/2}] \tag{2.3}$$

The Dirichlet Laplacian Δ^N on $l^2(V_q^N)$ is the bounded self-adjoint operator represented by the $|V_q^N| \times |V_q^N|$ submatrix of (2.2) obtained by restricting (2.2) to $k \leqslant N$ and $l \leqslant N$. The main result of this section is the following:

Theorem 2.1. The spectrum of $-\Delta^N$ is the discrete subset of $\mathbb{R}$ given by $\{\lambda_{n,k;N} \mid n = 1,\dots, k;\ k = 1,\dots, N+1\}$, where

$$\lambda_{n,k;N} = q - 2(q-1)^{1/2} \cos \frac{n\pi}{k+1} \qquad (n = 1,\dots, k;\ k = 1,\dots, N) \tag{2.4}$$

and

$$\lambda_{n,N+1;N} = q - 2(q-1)^{1/2} \cos \theta_n/2 \tag{2.5}$$

where θ_n is the solution of

$$\sin\left(1 + \frac{N}{2}\right)\theta_n = \frac{1}{q-1} \sin\left(\frac{N\theta_n}{2}\right) \tag{2.6}$$

with

$$0 < \theta_1 < \cdots < \theta_{N+1} < 2\pi \tag{2.7}$$

The corresponding multiplicities are given by

$$m_k = \begin{cases} q(q-2)(q-1)^{N-k-1} & \text{if} \quad 1 \leqslant k \leqslant N-1 \\ q-1 & \text{if} \quad k = N \\ 1 & \text{if} \quad k = N+1 \end{cases} \tag{2.8}$$

In this section we shall prove this theorem by a direct iteration argument. In the next section we give an alternative derivation (for a Cayley tree with root) using a random walk representation of the free fermion gas with Laplacian Δ^N.

We prove Theorem 2.1 here by the usual method of putting $\det(\lambda \mathbf{1} - \Delta^N) = 0$. For iteration purposes we generalize this determinant and define the matrix

$$M_N(\lambda_0, ..., \lambda_N; \mu_1, ..., \mu_N)_{(k; i_1, ..., i_k), (l; j_1, ..., j_l)}$$
$$= \lambda_k \delta_{k,l} \prod_{r=1}^{k} \delta_{i_r, j_r} + \mu_k \delta_{k,l+1} \prod_{r=1}^{l} \delta_{i_r, j_r} - \delta_{k+1,l} \prod_{r=1}^{k} \delta_{i_r, j_r} \tag{2.9}$$

with determinant

$$D_N(\lambda_0, ..., \lambda_N; \mu_1, ..., \mu_N) = \det M_N(\lambda_0, ..., \lambda_N; \mu_1, ..., \mu_N) \tag{2.10}$$

We then have the following result.

Lemma 2.1. The determinant (2.10) is given by

$$D_N(\lambda_0, ..., \lambda_N; \mu_1, ..., \mu_N)$$
$$= \prod_{k=1}^{N+1} \{f_k^{(N)}(\lambda_{N-k+1}, ..., \lambda_N; \mu_{N-k+2}, ..., \mu_N)\}^{m_k} \tag{2.11}$$

where the functions $f_k^{(N)}$ are defined by

$$f_0^{(N)} = 1, \qquad f_1^{(N)}(\lambda) = \lambda \tag{2.12a}$$

$$f_{k+1}^{(N)}(\lambda_0, ..., \lambda_k; \mu_1, ..., \mu_k) = \lambda_0 f_k^{(N)}(\lambda_1, ..., \lambda_k; \mu_2, ..., \mu_k)$$
$$+ (q-1)\, \mu_1 f_{k-1}^{(N)}(\lambda_2, ..., \lambda_k; \mu_3, ..., \mu_k) \tag{2.12b}$$

and

$$f_{N+1}^{(N)}(\lambda_0, ..., \lambda_N; \mu_1, ..., \mu_N) = \lambda_0 f_N^{(N)}(\lambda_1, ..., \lambda_N; \mu_2, ..., \mu_N)$$
$$+ q\mu_1 f_{N-1}^{(N)}(\lambda_2, ..., \lambda_N; \mu_3, ..., \mu_N) \tag{2.12c}$$

[The multiplicities m_k are given by (2.8)].

Proof. We use induction on N. For $N = 0$ we have obviously $D_0(\lambda_0) = \lambda_0 = f_1^{(0)}(\lambda_0)$. For $N = 1$ we can compute D_1 as follows. We multiply the first row ($k = 0$) by λ_1 and then add to it the q rows with $k = 1$. This yields

$$\lambda_1 D_1(\lambda_0, \lambda_1; \mu_1) = \lambda_1^q(\lambda_0 \lambda_1 + q\mu_1) = \lambda_1 f_1^{(1)}(\lambda_1)^{m_1} f_2^{(1)}(\lambda_0, \lambda_1; \mu_1)$$

The induction step proceeds similarly. We multiply the rows with $k = N - 1$ by λ_N and next add to each row with index $(N-1; i_1, ..., i_{N-1})$ the $q - 1$ rows with indices $(N; i_1, ..., i_{N-1}, j)$, where $j = 1, ..., q - 1$. In the resulting determinant the -1's in the columns with index $(N; i_1,..., i_{N-1}, j)$, $j=1,...,q-1$, have been replaced by zeros and the diagonal entries in columns $(N-1; i_1,..., i_{N-1})$ have changed to

$$\tilde{\lambda}_{N-1} = \lambda_{N-1}\lambda_N + (q-1)\,\mu_N \tag{2.13a}$$

while the entries μ_{N-1} in the columns $(N-2; i_1,..., i_{N-2})$ have changed to

$$\tilde{\mu}_{N-1} = \lambda_N \mu_{N-1} \tag{2.13b}$$

We conclude that, for $N \geqslant 2$,

$$\begin{aligned}&\lambda_N^{q(q-1)^{N-2}} D_N(\lambda_0,..., \lambda_N; \mu_1,..., \mu_N)\\ &\quad = \lambda_N^{q(q-1)^{N-1}} D_{N-1}(\lambda_0,..., \lambda_{N-2}, \tilde{\lambda}_{N-1}; \mu_1,..., \mu_{N-2}, \tilde{\mu}_{N-1})\end{aligned}$$

The lemma now follows from the observation that for $N \geqslant 2$ and $1 \leqslant k \leqslant N$,

$$\begin{aligned}&f_k^{(N-1)}(\lambda_0,..., \lambda_{k-2}, \tilde{\lambda}_{k-1}; \mu_1,..., \mu_{k-2}, \tilde{\mu}_{k-1})\\ &\quad = f_{k+1}^{(N)}(\lambda_0,..., \lambda_k; \mu_1,..., \mu_k)\end{aligned} \tag{2.14}$$

and the fact that $m_1 = q(q-2)(q-1)^{N-2} = q(q-1)^{N-1} - q(q-1)^{N-2}$. ∎

Proof of Theorem 2.1. It follows from Lemma 2.1 that the eigenvalues of $-\Delta^N$ are given by $q-$ the zeros of the functions $f_k(\lambda)$ defined recursively by

$$f_0^{(N)} = 1, \qquad f_1^{(N)}(\lambda) = \lambda \tag{2.15a}$$

$$f_{k+1}^{(N)}(\lambda) = \lambda f_k^{(N)}(\lambda) - (q-1)\, f_{k-1}^{(N)} \qquad \text{for} \quad 1 \leqslant k \leqslant N \tag{2.15b}$$

and

$$f_{N+1}^{(N)}(\lambda) = \lambda f_N^{(N)}(\lambda) - q f_{N-1}^{(N)}(\lambda) \tag{2.15c}$$

They occur with multiplicities given by m_k. Notice that $f_k(\lambda)$ is of order k, hence has k zeros, and

$$\sum_{k=1}^{N+1} k m_k = \frac{q(q-1)^N - 2}{q-2} = |V_q^N| \tag{2.16}$$

as it should be. It remains to show that the zeros of $f_k(\lambda)$ are of the form $q - \lambda_{n,k;N}$, where $\lambda_{n,k;N}$ is of the form (2.4), respectively (2.5). This can be seen by transforming to new variables; if we define

$$\tilde{f}_k(x) = (q-1)^{-k/2} f_k(2(q-1)^{1/2}\, x) \tag{2.17}$$

then the transformed functions satisfy (for $k \leqslant N-1$)

$$\tilde{f}_0(x)=1; \qquad \tilde{f}_1(x)=2x; \qquad \tilde{f}_{k+1}(x)=2x\tilde{f}_k(x)-\tilde{f}_{k-1}(x) \tag{2.18}$$

One recognizes the Chebyshev polynomials of the second kind[10]:

$$\tilde{f}_k(\cos\theta)=\frac{\sin(k+1)\theta}{\sin\theta} \tag{2.19}$$

for $k \leqslant N$. There is an exception for $k=N+1$, in which case

$$\tilde{f}_{N+1}(\cos\theta)=2\cos\theta\,\frac{\sin(N+1)\theta}{\sin\theta}-\frac{q}{q-1}\,\frac{\sin N\theta}{\sin\theta} \tag{2.20}$$

Changing θ to $\theta/2$, one obtains the relation (2.6). ∎

3. RANDOM WALK EVALUATION OF THE SPECTRUM

This section is devoted to an alternative derivation of the spectrum of the Dirichlet Laplacian on the Caley tree. It is based on the Feynman–Kac formula for the free fermion gas. Using this formula, we shall obtain a random walk formula for the partition function of the free fermion gas:

$$\ln Z_f(\beta,\mu)=\sum m_i \ln(1+e^{\beta(\mu-\lambda_i)})$$

from which we can read off the spectrum $\{\lambda_i\}$ and the multiplicities m_i. In Section 6 we shall show that this method can also be used to obtain the Bethe approximation to the spectrum and pressure of the free boson lattice gas.

The grand canonical partition function of the free fermion gas is given by

$$Z(\beta,\mu)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{x_1,\ldots,x_n\in\Gamma}\sum_{\pi\in S_n}\operatorname{sgn}(\pi) \times \langle x_{\pi(1)},\ldots,x_{\pi(n)}|\,e^{\beta(\mu n-\mathscr{H}_n)}\,|x_1,\ldots,x_n\rangle \tag{3.1}$$

Here Γ is an arbitrary finite lattice, the angle brackets denote the inner product in $l^2(\Gamma^n)$, $|x_1,\ldots,x_n\rangle$ is the state $\delta_{x_1}\otimes\cdots\otimes\delta_{x_n}$, and

$$\mathscr{H}_n=-\sum_{k=1}^{n}\Delta^{(k)} \qquad \text{with} \quad \Delta^{(k)}=\mathbf{1}\otimes\cdots\otimes\Delta\otimes\cdots\otimes\mathbf{1} \tag{3.2}$$

Now $-\mathscr{H}_n$ is the generator of n independent random walks, so that

$$\langle x_{\pi(1)},\ldots,x_{\pi(n)}|\,e^{-\beta\mathscr{H}_n}\,|x_1,\ldots,x_n\rangle=\mathbb{P}_{x_1,\ldots,x_n}(\xi_i(\beta)=x_{\pi(i)}\,\forall i) \tag{3.3}$$

Using the Markov property of random walks, it follows that if two random walks starting at x_i and x_j meet, then the corresponding probability equals the probability with indices i and j interchanged. We may therefore assume that no two random walks intersect and obtain

$$Z(\beta, \mu) = \sum_{n=0}^{\infty} \frac{e^{\beta\mu n}}{n!} \sum_{x_1,\dots,x_n \in \Gamma} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \times \mathbb{P}_{x_1,\dots,x_n}(\tau_c > \beta, \xi_i(\beta) = x_{\pi(i)} \forall i) \tag{3.4}$$

where τ_c is the first collision time. Next we approximate the continuous-time random walks by discrete-time random walks. We construct an augmented lattice $\mathscr{L}$ consisting of M copies of the lattice Γ and consider random walks of the following kind: A random path consists of a sequence of links $b_1, b_2, \dots$, where each link $b_i = ((x_i, m_i), (x_i', m_i'))$ satisfies $m_i' = m_i + 1$ and either x_i' and x_i are connected by a line in Γ or $x_i' = x_i$. (In any case, $x_{i+1} = x_i'$ and $m_{i+1} = m_i'$.) Links of the first type we call diagonal, those of the second kind vertical. To each diagonal link we ascribe a weight factor $\phi = \beta/M$ and to each vertical link a weight factor $\psi = 1 - q\beta/M$, where q is the coordinate number of the lattice. It is now easy to see that this discrete walk tends to the continuous-time Poisson random walk on Γ. Indeed,

$$\mathbb{P}_x(\xi_t = y) = \sum_{k=|y-x|}^{\infty} N_k(x \to y)\, p_k(t) \tag{3.5}$$

where $N_k(x \to y)$ is the number of paths in Γ from x to y, and $p_k(t)$ is the probability that there are k jumps in the time interval $(0, t)$. Similarly, we have for the discrete-time random walk,

$$\mathbb{P}_x^M(\xi_t = y) = \sum_{k=|y-x|}^{M} N_k(x \to y)\, p_k^M(t) \tag{3.6}$$

Now,

$$p_k(t) = \frac{t^k}{k!} e^{-t} \tag{3.7}$$

and

$$p_k^M(t) = (1 - \beta/M)^{[tM/\beta] - k} \binom{[tM/\beta]}{k} (\beta/M)^k \tag{3.8}$$

so that $p_k(t) = \lim_{M \to \infty} p_k^M(t)$ and hence

$$\mathbb{P}_x(\xi_t = y) = \lim_{M \to \infty} \mathbb{P}_x^M(\xi_t = y) \tag{3.9}$$

Using this result for each of the random walks in (3.4), we can write

$$Z(\beta, \mu) = \lim_{M \to \infty} \sum_{n=0}^{\infty} \frac{e^{\beta\mu n}}{n!} \sum_{x_1,\ldots,x_n} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \times \mathbb{P}^M_{x_1,\ldots,x_n}(\tau_c > \beta, \xi_i(\beta) = x_{\pi(i)} \forall i) \tag{3.10}$$

Next we can write the permutations π as a product of cycles $\pi = c_1 \cdots c_r$. The sign of π is then given by $\operatorname{sgn}(\pi) = \prod_{i=1}^r \operatorname{sgn}(c_i)$ and $\operatorname{sgn}(c_i) = (-1)^{|c_i|+1}$, where $|c_i|$ is the length of the cycle. The collection of paths ξ_i then decomposes into a collection of *closed* nonintersecting paths on the augmented lattice with periodic boundary conditions in the vertical direction. Thus we can write

$$Z(\beta, \mu) = \sum_{\gamma} \bar{\chi}(\gamma) \tag{3.11}$$

where the sum runs over all possible collections of closed, nonintersecting paths on the augmented lattice with periodic boundary conditions in the vertical direction, and

$$\bar{\chi}(\gamma) = \prod_{P \in \gamma} \bar{\omega}(P) \tag{3.12}$$

with

$$\bar{\omega}(P) = -\phi^{b_h} \psi^{b_v} \tag{3.13}$$

where b_h and b_v are the numbers of diagonal and vertical links, respectively, and ϕ and ψ are now given by

$$\begin{aligned} \phi &= \frac{\beta}{M} e^{(\beta\mu + i\pi)/M} \\ \psi &= \left(1 - \frac{q\beta}{M}\right) e^{(\beta\mu + i\pi)/M} \end{aligned} \tag{3.14}$$

[We have inserted a factor $e^{\beta\mu/M}$ in the weights ϕ and ψ to account for the factor $e^{\beta\mu n}$ in $Z(\beta, \mu)$. Notice that $\prod_{i=1}^r \operatorname{sgn}(c_i) = (-1)^{n+r}$. The factor $(-1)^r$ is given by the $-$ sign in (3.13); the factor $(-1)^n$ is given by the factors $e^{i\pi/M}$ in the definition of ϕ and ψ.]

The expression (3.11) can be evaluated along the lines of ref. 9 using the following graph-theoretic result, proved in refs. 5 and 6:

Theorem. The following equality holds:

$$\sum_{\gamma} \bar{\chi}(\gamma) = \prod_{P}{}' \, [1 + \bar{\omega}(P)] \tag{3.15}$$

where the sum runs over all collections of closed nonintersecting paths γ and the product is taken over all possible nonperiodic closed paths P.

It follows from this theorem[9] that $\ln Z(\beta, \mu)$ is given by

$$\ln Z(\beta, \mu) = -\sum_P{}' \sum_{j=1}^{\infty} \frac{[-\bar{\omega}(P)]^j}{j} = -\sum_{z \in \mathscr{L}} \sum_b \frac{\omega(P(z, b))}{b} \tag{3.16}$$

where $\omega(P(z, b))$ is the weighted sum over all possible closed paths of b bonds starting and ending at the site $z = (x, m) \in \mathscr{L}$. In the last identity of (3.16) we have used the fact that the product $[\bar{\omega}(P)]^j$ of j identical nonperiodic paths P can be interpreted as the weight of a path of periodicity j. Thus, the evaluation of the partition function of discrete Fermi trajectories on $\mathscr{L}$ is reduced to finding the sum $\sum_b [\omega(P(z, b))/b]$. The latter sum can be expressed easily through a generating function of random walks on the lattice Γ. The *generating function* $W(x, x_0)$ of random walks between sites x_0 and x ($x_0, x \in \Gamma$) is defined recursively by

$$\begin{aligned} W_0(x, x_0) &= \delta_{x, x_0} \\ W_j(x, x_0) &= \psi W_{j-1}(x, x_0) + \phi \sum_{|x'-x|=1} W_{j-1}(x', x_0) \end{aligned} \tag{3.17}$$

and

$$W(x, x_0) = \sum_{j=0}^{\infty} W_j(x, x_0) \tag{3.18}$$

$W(x, x_0)$ represents the probability that the walk starting at x_0 will arrive at x after an arbitrary number of steps. It is easily seen that the sum $\sum_b [\omega(P(z, b))/b]$ can be obtained from $W(x, x)$ by two successive transformations: $W(x, x) \to \tilde{W}(x, x) \to \tilde{\tilde{W}}(x, x)$. The first transformation divides each term of $W(x, x)$ by the number of steps:

$$\tilde{W}(x, x) = \sum_{j \geq 1} \frac{W_j(x, x)}{j} = \int_0^1 \frac{W^t(x, x) - 1}{t} \, dt \tag{3.19}$$

where W^t is obtained from W by replacing ϕ and ψ by $t\phi$ and $t\psi$, respectively. The second selects the paths of length $j = Mk$, where $k \geq 1$ is an integer. Using the identity

$$\frac{1}{M} \sum_{p=1}^{M} e^{2\pi i j p/M} = \sum_{k \in \mathbb{Z}} \delta_{j, kM}$$

we get

$$\tilde{\tilde{W}}(x, x) = \frac{1}{M} \sum_{p=1}^{M} \tilde{W}^{\alpha}(x, x) \qquad \text{with} \quad \alpha = e^{2\pi i p/M} \tag{3.20}$$

We have

$$\sum_{b} \frac{\omega(P(z, b))}{b} = \tilde{\tilde{W}}(x, x) \tag{3.21}$$

where x is the first coordinate of the site $z = (x, m)$.

The generating function of walks on the Cayley tree is related to the generating function of one-dimensional walks on the interval $[0, N]$ as follows (for simplicity we consider a Cayley tree with root, i.e., the point 0 is connected to $q-1$ others, instead of the homogeneous Cayley tree considered in Section 2; this does not change the formula for the pressure in the infinite-volume limit): The generating function for a biased one-dimensional walk on $[0, N]$ with weights given by $(q-1)\phi$ for steps from n to $n+1$ and ϕ for steps from n to $n-1$ and ψ for staying at the point n in the case of closed walks is the same as that of an unbiased walk with weights $\phi(q-1)^{1/2}$ and ψ, which is given by [see ref. 11, Chapter V; $W_N(n) = g_N(n, n)$]

$$W_N(n) = \frac{2}{N+2} \sum_{k=0}^{N} \frac{\sin^2[(k+1)(n+1)\pi/(N+2)]}{1 - \psi - 2\phi(q-1)^{1/2} \cos[(k+1)\pi/(N+2)]} \tag{3.22}$$

If now $W_N(n, l)$ is the generating function of the subset of one-dimensional random walks on $[0, N]$ with $\min(n) = l$, then

$$W_N(n, l) = W_{N-l}(n-l) - W_{N-l-1}(n-l-1) \tag{3.23}$$

while each walk contributing to $W_N(n, l)$ corresponds to $(q-1)^{n-l}$ walks on the Cayley tree contributing to $W(x, x)$. Hence

$$W(x, x) = \begin{cases} \displaystyle\sum_{l=0}^{n-1} (q-1)^{-n+l} [W_{N-l}(n-l) - W_{N-l-1}(n-l-1)] & \\ \qquad + W_{N-n}(0) & \text{for} \quad n = 1, 2, \ldots, N \\ W_N(0) & \text{for} \quad n = 0 \end{cases} \tag{3.24}$$

Now we must perform the transformations $W \to \tilde{W} \to \tilde{\tilde{W}}$. The first yields

$$\begin{aligned} \tilde{W}_N(x, x) &= \int_0^1 \frac{dt}{t} \left[\frac{2}{N+2} \sum_{k=0}^{N} (1 - t\lambda_k)^{-1} \sin^2 \frac{k+1}{N+2} \pi(n+1) - 1 \right] \\ &= -\frac{2}{N+2} \sum_{k=0}^{N} \sin^2 \left(\frac{k+1}{N+2} \pi(n+1) \right) \ln(1 - \lambda_k) \end{aligned} \tag{3.25}$$

We have put

$$\lambda_k = \psi + 2\phi(q-1)^{1/2} \cos \frac{k+1}{N+2} \pi$$

Inserting (3.14), we have

$$[\psi + 2\phi(q-1)^{1/2} \cos \theta]^M$$
$$\approx -\exp\{\beta[\mu - q + 2(q-1)^{1/2} \cos \theta]\} \qquad \text{as} \quad M \to \infty \quad (3.26)$$

Thus we obtain

$$\tilde{\tilde{W}}_N(x, x) = -\frac{2}{M(N+2)} \sum_{k=0}^{N} \sin^2 \left[\frac{k+1}{N+2} \pi(n+1)\right]$$
$$\times \ln\left(1 + \exp\left\{\beta\left[\mu - \lambda\left(\frac{k+1}{N+2}\right)\right]\right\}\right) \qquad (3.27)$$

where we have written

$$\lambda\left(\frac{m}{n}\right) = q - 2(q-1)^{1/2} \cos \frac{m\pi}{n} \qquad (3.28)$$

Taking into account that the number of lattice points with coordinate n is $(q-1)^n$, we can write Eq. (3.22) in the form

$$\ln Z(\beta, \mu) = -M \sum_{n=0}^{N} (q-1)^n \tilde{\tilde{W}}(x, x) \qquad (3.29)$$

where $\tilde{\tilde{W}}(x, x)$ follows on substitution of (3.27) into (3.24). We can now obtain an explicit form for $\ln Z(\beta, \mu)$ by a straightforward but tedious calculation:

$$\ln Z(\beta, \mu) = -M \sum_{n=1}^{N} (q-1)^n \left\{ \tilde{\tilde{W}}_{N-n}(0) \right.$$
$$\left. + \sum_{l=0}^{n-1} \frac{\tilde{\tilde{W}}_{N-l}(n-l) - \tilde{\tilde{W}}_{N-l-1}(n-l-1)}{(q-1)^{n-l}} \right\} - M\tilde{\tilde{W}}_N(0)$$
$$= -M(q-1)^N \left\{ \sum_{k=0}^{N} \frac{\tilde{\tilde{W}}_k(0)}{(q-1)^k} \right.$$
$$\left. + \sum_{k=1}^{N} \sum_{m=1}^{k} \frac{\tilde{\tilde{W}}_k(m) - \tilde{\tilde{W}}_{k-1}(m-1)}{(q-1)^k} \right\} \qquad (3.30)$$

Using (3.27) and the identity

$$\frac{2}{N+2}\sum_{m=0}^{k}\sin^2\frac{p+1}{k+2}\pi(m+1)=1 \tag{3.31}$$

we have

$$\begin{aligned}
\frac{\ln Z(\beta,\mu)}{(q-1)^N}&=\sum_{k=0}^{N}\frac{1}{(q-1)^k}\frac{1}{k+2}\sum_{p=0}^{k}2\sin^2\left(\frac{p+1}{k+2}\pi\right)\\
&\quad\times\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{k+2}\right)\right]\right\}\\
&\quad+\sum_{k=1}^{N}\sum_{m=1}^{k}\frac{1}{(q-1)^k}\\
&\quad\times\left(\frac{2}{k+2}\sum_{p=0}^{k}\sin^2\left[\frac{p+1}{k+2}\pi(m+1)\right]\right.\\
&\quad\times\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{k+2}\right)\right]\right\}\\
&\quad-\frac{2}{k+1}\sum_{p=0}^{k-1}\sin^2\left(\frac{p+1}{k+1}\pi m\right)\\
&\quad\left.\times\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{k+1}\right)\right]\right\}\right)\\
&=\ln[1+\exp\beta(\mu-q)]+\sum_{k=1}^{N}\frac{1}{(q-1)^k}\\
&\quad\times\left(\sum_{p=0}^{k}\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{k+2}\right)\right]\right\}\right.\\
&\quad\left.-\sum_{p=0}^{k-1}\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{k+1}\right)\right]\right\}\right)
\end{aligned}\tag{3.32}$$

The second sum in the final expression can be expressed in terms of the first:

$$\begin{aligned}
S_2&=\frac{S_1}{q-1}+\frac{1}{q-1}\ln[1+\exp\beta(\mu-q)]\\
&\quad-\frac{1}{(q-1)^{N+1}}\sum_{p=0}^{N}\ln\left\{1+\exp\beta\left[\mu-\lambda\left(\frac{p+1}{N+2}\right)\right]\right\}
\end{aligned}\tag{3.33}$$

We obtain finally

$$\ln Z(\beta, \mu) = (q-2)(q-1)^N \sum_{k=1}^{N+1} \frac{1}{(q-1)^k} \times \sum_{m=1}^{k} \ln \left\{1 + \exp \beta \left[\mu - \lambda\left(\frac{m}{k+1}\right)\right]\right\} + \frac{1}{q-1} \sum_{m=1}^{N+1} \ln \left\{1 + \exp \beta \left[\mu - \lambda\left(\frac{m}{N+2}\right)\right]\right\} \tag{3.34}$$

We can now read off the spectrum of $-\Delta^N$ on the Cayley tree with root: it has the form

$$\lambda_{m,k} = q - 2(q-1)^{1/2} \cos \frac{m\pi}{k+1}, \qquad k = 1,..., N+1, \quad m = 1,..., k \tag{3.35}$$

with multiplicities

$$\begin{aligned} m'_k &= (q-2)(q-1)^{N-k} \qquad \text{for} \quad k = 1,..., N \\ m'_{N+1} &= 1 \end{aligned} \tag{3.36}$$

Notice that only $\lambda_{n,N+1}$ differ from the eigenvalues computed in Theorem 2.1 for the homogeneous Cayley tree. The spectrum (3.35) can also be derived by the method of Section 2.

4. EXISTENCE OF CONDENSATION IN THE THERMODYNAMIC LIMIT

In this section we prove the existence of the thermodynamic limit for the free boson gas on the Cayley tree. Next we prove that this gas exhibits generalized boson condensation in the sense introduced by Girardeau.[4] In the following section we shall prove that the condensation is actually into the ground state only, i.e., $\rho_c = \rho_m$ in the sense of ref. 2.

We denote the finite-volume ground state by $\lambda_1(N)$, i.e.,

$$\lambda_1(N) = \min\{\lambda_{n,k;N} \mid n = 1,..., k; k = 1,..., N+1\} \tag{4.1}$$

It follows from Theorem 2.1 and Theorem 3.1 that for both Cayley trees considered,

$$\lim_{N \to \infty} \lambda_1(N) = q - 2(q-1)^{1/2} \tag{4.2}$$

Next we define the single-particle partition function as in ref. 2:

$$\phi_N(\beta) = |V_q^N|^{-1} \operatorname{trace} \exp\{\beta[\Delta^N + \lambda_1(N)\mathbf{1}]\} \tag{4.3}$$

We shall define the pressure $p_N(\mu)$ by

$$\exp[\beta\, |V_q^N|\, p_N(\mu)] = \sum_{n=0}^{\infty} e^{n\beta\mu} \sum_{\{n(k)\geqslant 0 \mid \sum n(k)=n\}} \times \exp\left\{-\beta \sum_{k=1}^{\infty} n(k)[\lambda_k(N) - \lambda_1(N)]\right\} \tag{4.4}$$

where $\lambda_1(N) < \lambda_2(N) \leqslant \lambda_3(N) \leqslant \cdots$ are the eigenvalues of $-\Delta^N$ in ascending order and including multiplicities. [By subtracting the ground-state eigenvalue we have renormalized the chemical potential; thus, the thermodynamic limit is defined for $\mu \in (-\infty, 0)$.] It was proven in ref. 2 that whenever the limit $\phi(\beta) = \lim_{N\to\infty} \phi_N(\beta)$ exists for all $\beta \in (0, \infty)$ and is nonzero for some $\beta \in (0, \infty)$, then:

1. The limit $p(\mu) = \lim_{N\to\infty} p_N(\mu)$ exists for all $\mu \in (-\infty, 0)$ and is given by

$$\beta p(\mu) = \int_{[0,\infty)} \ln(1 - e^{\beta(\mu-\lambda)})^{-1}\, dF(\lambda) \tag{4.5}$$

where the density of states $dF(\lambda)$ is defined by

$$\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda}\, dF(\lambda) \tag{4.6}$$

2. If $\mu_N(\rho)$ is the unique root of the equation

$$\frac{d}{d\mu} p_N(\mu) = \rho \tag{4.7}$$

then the limit $\mu(\rho) = \lim_{N\to\infty} \mu_N(\rho)$ exists for all $\rho \in (0, \infty)$; it is the unique root in $(-\infty, 0)$ of the equation $(d/d\mu)\, p(\mu) = \rho$ for $\rho < \rho_c$ and $\mu(\rho) = 0$ for $\rho \geqslant \rho_c$.

Here, the critical density ρ_c is given by

$$\rho_c = \int_{[0,\infty)} \frac{dF(\lambda)}{e^{\beta\lambda} - 1} \tag{4.8}$$

3. The limit $\pi(\rho) = \lim_{N\to\infty} \pi_N(\rho)$ exists and is given by $\pi(\rho) = (p \circ \mu)(\rho)$.

4. The double limit

$$v_0(\rho) = \lim_{\lambda\downarrow 0} \lim_{N\to\infty} \mathbb{E}_\rho[|V_q^N|^{-1} X_N(\lambda)] \tag{4.9}$$

exists and is given by

$$v_0(\rho) = (\rho - \rho_c)^+ \tag{4.10}$$

where $(x)^+$ stands for the positive part of x. Here $\mathbb{E}_\rho$ denotes the grand-canonical expectation at mean density ρ, and $X_N(\lambda) = \sum_{k:\,\lambda_N(k)<\lambda} n(k)$.

We say that generalized condensation occurs if $v_0(\rho) > 0$ for some ρ, i.e., if $\rho_c < \infty$. We shall now prove that the condition on the existence of $\phi(\beta)$ is satisfied in our model.

Lemma 4.1. For $\beta > 0$ and $q = 3, 4,\ldots$, the limit $\phi(\beta) = \lim_{N\to\infty} \phi_N(\beta)$ exists and is given by

$$\phi(\beta) = (q-2)^2 \sum_{k=1}^{\infty} \sum_{n=1}^{k} (q-1)^{-k} \times \exp\left[-4\beta(q-1)^{1/2} \sin^2 \frac{n\pi}{2k+2}\right] \tag{4.11}$$

Proof. By Theorem 2.1 and formula (2.1),

$$\phi_N(\beta) = \frac{q-2}{q(q-1)^N - 2} \left\{ \sum_{k=1}^{N} \sum_{n=1}^{k} m_k e^{-\beta(\lambda_{n,k;N} - \lambda_1(N))} + \sum_{n=1}^{N+1} e^{-\beta(\lambda_{n,N+1;N} - \lambda_1(N))} \right\} \tag{4.12}$$

But

$$\lim_{N\to\infty} \frac{q-2}{q(q-1)^N - 2} \sum_{n=1}^{N+1} e^{-\beta(\lambda_{n,N+1;N} - \lambda_1(N))} = 0$$

and

$$\lim_{N\to\infty} \frac{q-2}{q(q-1)^N - 2} \sum_{n=1}^{N} e^{-\beta(\lambda_{n,N;N} - \lambda_1(N))} = 0$$

so that

$$\phi(\beta)=\lim_{N\to\infty}\sum_{k=1}^{N-1}\sum_{n=1}^{k}\frac{q(q-2)^2\,(q-1)^{N-k-1}}{q(q-1)^N-2}$$
$$\times\exp\{-\beta[\lambda_{n,k;N}-\lambda_1(N)]\}$$
$$=\lim_{N\to\infty}(q-2)^2\sum_{k=1}^{N-1}\sum_{n=1}^{k}(q-1)^{-k-1}$$
$$\times\exp\{-\beta[\lambda_{n,k;N}-\lambda_1(N)]\}$$
$$=(q-2)^2\sum_{k=1}^{\infty}\sum_{n=1}^{k}(q-1)^{-k-1}$$
$$\times\exp\left[-4\beta(q-1)^{1/2}\sin^2\left(\frac{n\pi}{2k+2}\right)\right]\quad\blacksquare\tag{4.13}$$

To prove that condensation occurs, it is now sufficient to show that $\rho_c<\infty$. This is the content of the following lemma:

Lemma 4.2. For $\beta>0$ and $q=3, 4,...,$

$$\rho_c=\sum_{l=1}^{\infty}\phi(l\beta)\leqslant\frac{\pi^2(4q^2-11q+8)}{24\beta(q-1)^{3/2}\,(q-2)}\tag{4.14}$$

Proof. By Lemma 4.1,

$$\sum_{l=1}^{\infty}\phi(l\beta)=(q-2)^2\sum_{k=1}^{\infty}\sum_{n=1}^{k}(q-1)^{-k-1}$$
$$\times\left\{\exp\left[4\beta(q-1)^{1/2}\sin^2\left(\frac{n\pi}{2k+2}\right)\right]-1\right\}^{-1}$$
$$\leqslant(q-2)^2\sum_{k=1}^{\infty}\sum_{n=1}^{k}(q-1)^{-k-1}$$
$$\times\left[4\beta(q-1)^{1/2}\sin^2\left(\frac{n\pi}{2k+2}\right)\right]^{-1}$$
$$\leqslant(q-2)^2\sum_{k=1}^{\infty}\sum_{n=1}^{k}(q-1)^{-k-3/2}\,(k+1)^2\,n^{-2}(4\beta)^{-1}$$
$$\leqslant(q-2)^2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(q-1)^{-k-3/2}\,(k+1)^2\,n^{-2}(4\beta)^{-1}$$
$$=\frac{(4q^2-11q+8)}{(q-1)^{3/2}\,(q-2)}\frac{\pi^2}{24\beta}\quad\blacksquare\tag{4.15}$$

Corollary. The boson lattice gas on a Cayley tree with coordination number $q \geqslant 3$ exhibits generalized boson condensation.

5. CONDENSATION INTO THE GROUND STATE

In order to prove that the condensation is in fact entirely into the ground state, we must estimate the distance between the ground state and the first excited state. For the ground state we have the following result.

Lemma 5.1. For $N \geqslant 2$ and $q \geqslant 3$,

$$\lambda_1(N) = \lambda_{1, N+1; N} \tag{5.1}$$

and the following estimate holds:

$$\lambda_1(N) \leqslant q - 2(q-1)^{1/2} \cos\left(\frac{\pi}{N+2}\right) \tag{5.2}$$

Proof. It is clearly sufficient to prove the estimate (5.2). We must show that Eq. (2.6) has a solution in the interval $(0, 2\pi/(N+2)]$. To this end, we define

$$g(\theta) = \sin\left(1 + \frac{N}{2}\right)\theta - \frac{1}{q-1} \sin\left(\frac{N\theta}{2}\right) \tag{5.3}$$

Then $g(0) = 0$ and $g'(0) > 0$, while

$$g\left(\frac{2\pi}{N+2}\right) = -\frac{1}{q-1} \sin\left(\frac{N\pi}{N+2}\right) < 0$$

This implies that $g(\theta)$ has a zero in the interval $(0, 2\pi/(N+2)]$. ∎

Similarly, for the first excited state we have the following result.

Lemma 5.2. For $N \geqslant 2$ the first excited state is given by

$$\lambda_2(N) = \lambda_{1, N; N} = q - 2(q-1)^{1/2} \cos\left(\frac{\pi}{N+1}\right) \tag{5.4}$$

Proof. We must show that $\lambda_{1,N;N} < \lambda_{2,N+1;N}$, or equivalently, $\theta_2 > 2\pi/(N+1)$. Now, for $\theta \leqslant \pi/(N+2)$,

$$g(\theta) > \frac{N+2}{\pi}\theta - \frac{1}{q-1}\frac{N\theta}{2} > 0$$

On the other hand, for $\pi/(N+2)<\theta<\pi/N$, $\cos(1+N/2)\theta<0$, while $\cos(N\theta/2)>0$, so that $g'(\theta)<0$. Moreover, for $\pi/N<\theta<2\pi/(N+2)$, $\cos(1+N/2)\theta<\cos(N\theta/2)<0$ because the function $\cos x$ is decreasing on $(\pi/2, \pi)$. It follows that $g'(\theta)<0$ for all $\theta\in(\pi/(N+2), 2\pi/(N+2))$, so that g has a unique zero in this interval. This implies that $\theta_2>2\pi/(N+2)$. Finally, for $2\pi/(N+2)\leqslant\theta\leqslant 2\pi/N$, $\sin(1+N/2)\theta>0$, while $\sin(N\theta/2)<0$, so that $g(\theta)$ has no zero in this interval and $\theta_2>2\pi/N>2\pi/(N+1)$. ∎

Corollary. For $N\geqslant 2$,

$$\lambda_2(N)-\lambda_1(N)\geqslant(N+1)^{-3} \tag{5.5}$$

Proof. By (5.2) and (5.4),

$$\begin{aligned}
\lambda_2(N)-\lambda_1(N)&\geqslant 2(q-1)^{1/2}\left[\cos\left(\frac{\pi}{N+2}\right)-\cos\left(\frac{\pi}{N+1}\right)\right]\\
&=4(q-1)^{1/2}\sin\left(\frac{\pi}{2(N+1)(N+2)}\right)\\
&\quad\times\sin\left(\frac{(2N+3)\pi}{2(N+1)(N+2)}\right)\\
&\geqslant 4(q-1)^{1/2}\frac{2N+3}{(N+1)^2\,(N+2)^2}\\
&\geqslant\frac{1}{(N+1)^3} \quad ∎
\end{aligned} \tag{5.6}$$

As in ref. 2, we now define a rescaled single-particle partition function by

$$\gamma_N(\beta)=\text{trace}\exp\{\beta\,|V_q^N|\,[\Delta^N+\lambda_1(N)\mathbf{1}]\} \tag{5.7}$$

We then have the following result.

Lemma 5.3.

$$\lim_{N\to\infty}\gamma_N(\beta)=1 \tag{5.8}$$

Proof. Since $\lambda_1(N)$ is an eigenvalue of $-\Delta^N$, we have $\gamma_N(\beta)\geqslant 1$. To obtain the converse inequality, we express $\gamma_N(\beta)$ in terms of $\phi_N(\beta)$:

$$\gamma_N(\beta) = \sum_{k=1}^{\infty} \exp\{-\beta\, |V_q^N|\, [\lambda_k(N) - \lambda_1(N)]\}$$

$$= 1 + \sum_{k=2}^{\infty} \exp\{-\beta(|V_q^N| - 1)[\lambda_k(N) - \lambda_1(N)]$$

$$- \beta[\lambda_k(N) - \lambda_1(N)]\}$$

$$\leqslant 1 + \frac{1}{|V_q^N|} \exp\{-\beta(|V_q^N| - 1)[\lambda_2(N) - \lambda_1(N)]\}\, \phi_N(\beta) \tag{5.9}$$

But, by (5.5) and (2.1),

$$\lim_{N\to\infty} \frac{1}{|V_q^N|} \exp\{-\beta(|V_q^N| - 1)[\lambda_2(N) - \lambda_1(N)]\} = 0$$

so that $\limsup_{N\to\infty} \gamma_N(\beta) \leqslant 1$. ∎

We also need an estimate on the other eigenvalues:

Lemma 5.4. For $k = 1,..., N-1$ and $n = 1,..., k$ the following inequality holds:

$$\lambda_{n,k;N} - \lambda_1(N) \geqslant \frac{N-k}{Nk^2} \tag{5.10}$$

Proof. As in the proof of the Corollary of Lemma 5.2, we have

$$\lambda_{n,k;N} - \lambda_1(N) \geqslant \lambda_{1,k;N} - \lambda_2(N)$$

$$= 2(q-1)^{1/2} \left[\cos\left(\frac{\pi}{N+1}\right) - \cos\left(\frac{\pi}{k+1}\right)\right]$$

$$\geqslant 4(q-1)^{1/2} \frac{(N-k)(N+k+2)}{(N+1)^2\,(k+1)^2} \geqslant \frac{N-k}{Nk^2}$$ ∎

Lemma 5.5. Let the maximum density in levels above the ground state be defined by

$$\rho_m = \limsup_{N\to\infty} |V_q^N|^{-1} \sum_{k=2}^{\infty} (e^{\beta[\lambda_k(N) - \lambda_1(N)]} - 1)^{-1} \tag{5.11}$$

Then $\rho_m = \rho_c$.

Proof. By Lebesgue's dominated convergence theorem,

$$\rho_c = \lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} |V_q^N|^{-1}$$

$$\times \sum_{\{k\geqslant 2 \,|\, \lambda_k(N) - \lambda_1(N) \geqslant \varepsilon\}} (e^{\beta[\lambda_k(N) - \lambda_1(N)]} - 1)^{-1} \tag{5.12}$$

Hence it is sufficient to show that

$$\limsup_{\varepsilon\downarrow 0}\limsup_{N\to\infty}|V_q^N|^{-1} \times \sum_{\{k\geqslant 2\,|\,\lambda_k(N)-\lambda_1(N)<\varepsilon\}} (e^{\beta[\lambda_k(N)-\lambda_1(N)]}-1)^{-1}=0 \tag{5.13}$$

The contribution to this expression of the terms with $\lambda_k(N)=\lambda_{n,N;N}$ or $\lambda_k(N)=\lambda_{n,N+1;N}$ is bounded above by

$$|V_q^N|^{-1}\left\{\frac{N+1}{e^{\beta[\lambda_2(N)-\lambda_1(N)]}-1}+\frac{N(q-1)}{e^{\beta[\lambda_2(N)-\lambda_1(N)]}-1}\right\} \tag{5.14}$$

and hence tends to zero. The contribution of the remaining terms can be bounded with the help of Lemma 5.4 as follows:

$$\begin{aligned}
&|V_q^N|^{-1}\sum_{\{k\leqslant N-1\,|\,\lambda_{n,k;N}-\lambda_1(N)<\varepsilon\}}\sum_{n=1}^{k}\frac{q(q-2)(q-1)^{N-k-1}}{e^{\beta[\lambda_{n,k;N}-\lambda_1(N)]}-1}\\
&\quad\leqslant|V_q^N|^{-1}\sum_{\{k\leqslant N-1\,|\,(N-k)/(Nk^2)<\varepsilon\}}\frac{kq(q-2)(q-1)^{N-k-1}}{e^{\beta(N-k)/(Nk^2)}-1}\\
&\quad\leqslant\sum_{\{k\leqslant N-1\,|\,(N-k)/(Nk^2)<\varepsilon\}}\frac{kq(q-2)^2\,(q-1)^{N-k-1}}{q(q-1)^N-2}\,\frac{Nk^2}{\beta(N-k)}
\end{aligned} \tag{5.15}$$

Hence

$$\begin{aligned}
&\limsup_{N\to\infty}|V_q^N|^{-1}\sum_{\{k\leqslant N-1\,|\,\lambda_{n,k;N}-\lambda_1(N)<\varepsilon\}}\sum_{n=1}^{k}\frac{q(q-2)(q-1)^{N-k-1}}{e^{\beta[\lambda_{n,k;N}-\lambda_1(N)]}-1}\\
&\quad\leqslant(q-2)^2\,\beta^{-1}\sum_{\{k\,|\,k^{-2}<\varepsilon\}}(q-1)^{-k-1}\,k^3
\end{aligned} \tag{5.16}$$

The result now follows from the convergence of the latter series. ▮

Corollary. In the free boson lattice gas on a Cayley tree the condensation is entirely into the ground state, i.e.,

$$v_1(\rho)=v_0(\rho)=(\rho-\rho_c)^+ \tag{5.17}$$

6. THE BETHE APPROXIMATON

It is well known that, for the Ising model, the Bethe approximation to the free energy does not agree with the exact expression for this model on a Cayley tree.[3] In fact, the former exhibits a phase transition, while the latter does not. We have seen that in the case of the free boson lattice gas on a Cayley tree there is a phase transition. We shall see that there nevertheless is a difference between the exact model and the Bethe approximation, which manifests itself here in the fact that the latter turns out to have a continuous spectrum, as opposed to the exact model, which has a dense discrete spectrum. The Bethe approximation consists in an identification of all sites of the Bethe lattice which can be considered as an infinite Cayley tree. This means that, instead of the sum (3.16) over all lattice sites $z \in \mathscr{L}$, we must take only the term $\sum_b [\omega(P(0, b))/b]$, where $\omega(P(0, b))$ is the generating function of all possible closed paths of b bonds on the infinite lattice. In other words, we must chose in Eq. (3.24) the starting point n infinitely remote from the left and right side of the interval $[0, N]$ as $N \to \infty$. In this case (3.24) becomes

$$W(x, x) = W_\infty(0) + \sum_{l=1}^{\infty} (q-1)^{-l} [W_\infty(l) - W_\infty(l-1)] \tag{6.1}$$

where $W_\infty(l)$ is the generating function for one-dimensional closed walks starting at the site l on the semi-infinite lattice 0, 1, 2,... . The expression for $W_\infty(l)$ is a limit of (3.22):

$$W_\infty(l) = \frac{1}{\pi} \int_0^\pi \frac{2 \sin^2(l+1)\theta}{1 - \lambda(\theta)} \, d\theta \tag{6.2}$$

with

$$\lambda(\theta) = \psi + 2\phi(q-1)^{1/2} \cos\theta \tag{6.3}$$

Substituting into (6.1), we get

$$\begin{aligned} W(0, 0) &= \frac{2}{\pi} \sum_{l=0}^{\infty} \int_0^\pi \frac{\sin^2(l+1)\theta - \sin^2 l\theta}{(q-1)^l} \frac{d\theta}{1 - \lambda(\theta)} \\ &= \frac{2}{\pi} \int_0^\pi \frac{q(q-1)}{1-\lambda(\theta)} \frac{\sin^2\theta}{1 - 2(q-1)\cos 2\theta + (q-1)^2} \, d\theta \end{aligned} \tag{6.4}$$

where we have used the identity

$$\sum_{l=0}^{\infty} x^l \cos l\theta = \frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2} \tag{6.5}$$

The transformations $W \to \tilde{W} \to \tilde{\tilde{W}}$ are analogous to the ones performed in Section 3, so we simply state the result for the fermion gas pressure:

$$\begin{aligned} p_f(\beta) = \frac{2q(q-1)}{\pi\beta} \int_0^\pi & \frac{\sin^2\theta}{1 - 2(q-1)\cos 2\theta + (q-1)^2} \\ & \times \ln(1 + e^{\beta[\mu - q + 2(q-1)^{1/2}\cos\theta]}) \, d\theta \end{aligned} \tag{6.6}$$

We see from this formula that the spectrum is continuous, in contrast with the exact model. For the pressure of the boson gas in the Bethe approximation we infer

$$p(\beta) = -\frac{2q(q-1)}{\pi\beta} \int_0^\pi \frac{\sin^2\theta}{1-2(q-1)\cos 2\theta + (q-1)^2} \times \ln(1 - e^{\beta[\mu - 2(q-1)^{1/2}(1-\cos\theta)]})\, d\theta \tag{6.7}$$

where we have again renormalized the energy scale so that $\mu \in (-\infty, 0)$. It is easily seen that the density of states behaves as $dF(\lambda) \sim \lambda\, d\lambda$ as $\lambda \to 0$, so that

$$\rho_c = \frac{2q(q-1)}{\pi} \int_0^\pi \frac{\sin^2\theta}{1-2(q-1)\cos 2\theta + (q-1)^2} \times \frac{d\theta}{e^{2\beta(q-1)^{1/2}(1-\cos\theta)} - 1} < \infty \tag{6.8}$$

The Bethe approximation therefore also exhibits generalized condensation.

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